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Binary mixtures of Bose-Einstein condensates: Phase dynamics and spatial dynamics

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Abstract. We investigate the relative phase coherence properties and the occurrence of demixing instabilities for two mutually interacting and time evolving Bose-Einstein condensates in traps. Our treatment naturally includes the additional decoherence effect due to fluctuations in the total number of particles. Analytical results are presented for the breathe-together solution, an extension of previously known scaling solution to the case of a binary mixture of condensates. When the three coupling constants describing the elastic interactions among the atoms in the two states are close to each other, a dramatic increase of the phase coherence time is predicted. Numerical results are presented for the parameters of the recent JILA experiments.

PACS. 03.75.Fi Phase coherent atomic ensembles; quantum condensation phenomena – 05.30.Jp Boson systems

1 Introduction

Since the recent experimental observation of Bose-Einstein condensation in dilute atomic gases [1], much interest has been raised about the coherence properties of the condensates. Considerable attention has been devoted to the matter of the relative phase between two condensates: how this phase manifests itself in an interference experiment [2,3], how it can be established by measurement [4,5], and how it evolves in presence of atomic interactions [5–7] and in presence of particle losses [8].

As it was proved in recent experiments performed at JILA, binary mixtures of condensates represent an ideal system to study the phase coherence properties of Bose-Einstein condensates [9]. In these experiments two condensates in two different internal atomic states are created with a well-defined relative phase. After a time τ during which the condensates evolve in the trapping potentials, one mixes coherently the two internal atomic states which makes the two condensates interfere; from the spatial interference pattern one gets the relative phase of the two condensates. By repeating the whole experimental process, one has access to the distribution of the relative phase after an evolution time τ, so that one can investigate phase decoherence as function of time.

The interaction between the two condensates in the JILA experiment gives rise to a rich spatial separation dynamics between the two condensates [10], which complicates the theoretical study of the relative phase dynamics. As a consequence previous theoretical treatments of the phase decoherence processes, dealing essentially with steady state condensates, as in [11], cannot *a priori* be applied to the experimental situation.

A treatment of the phase coherence of two interacting, non stationary, condensates can be found in [12], with two important differences as compared to the present situation of interest: (1) in [12] the condensates are subject to a continuous coherent coupling of amplitude Λ; results are obtained from a perturbative expansion in powers of $1/\Lambda$ and cannot be simply extended to the present $\Lambda = 0$ case; (2) in [12] all the coupling constants $g_{\rm aa}, g_{\rm ab}, g_{\rm bb}$ between the two internal atomic states a and b are assumed to be equal.

In this paper we propose a formalism to study the relative phase dynamics of interacting and dynamically evolving Bose-Einstein condensates initially at zero temperature.

We present the general method in Section 2. It consists in expanding the initial state on Fock states, and in evolving each Fock state in the Hartree-Fock approximation. We determine the time dependence of the phase collapse for a binary mixture of condensates, due to (1) fluctuations in the relative number of particles between the condensates, intrinsic to the initial state with well-defined relative phase, and (2) fluctuations in the total number of particles. In the next two sections we apply this general formalism to two limiting cases that can be treated analytically.

The first case, in Section 3, considers a particular solution of the non-linear Schrödinger equations for the condensates wavefunctions; in this solution the

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two condensates remain spatially superimposed as they breathe in phase, provided that dynamical stability conditions (that we determine) are satisfied. We find that phase decoherence can be highly reduced with respect to non mutually interacting condensates when the three coupling constants $g_{\rm aa}, g_{\rm ab}, g_{\rm bb}$ between atoms in the two internal states a, b are close to each other.

In Section 4, we therefore study in a more general case (not restricted to the breathe-together solution) the dynamics of the relative phase for a mixture of condensates for close coupling constants. Our treatment requires also in this case the absence of demixing instability, a point that we discuss in detail.

Finally we discuss the case of the JILA experiment in Section 5. This case, that corresponds to close coupling constants in a regime of demixing instability, is more difficult to analyze. The predicted phase collapse time depends on the fluctuations of the total number of particles; it is on the order of 0.4 s for Gaussian fluctuations of 8%.

2 General method

In this section, we introduce a gedanken experiment to characterize phase coherence between two condensates: the relevant quantity is the interference term $\langle \hat{\psi}_{\rm b}{}^\dagger(\mathbf{r},t)\hat{\psi}_{\rm a}(\mathbf{r},t)\rangle$ between the atomic fields of the two condensates a and b. Subsequently we express this interference term in the Hartree-Fock approximation, assuming an initially well-defined relative phase between the condensates. After a further approximation on the modulus and the phase of the condensate wavefunctions, we determine the decay with time of the interference term due to atomic interactions; we arrive at the simple results equation (18) for a fixed total number of particles and equation (22) for Gaussian fluctuations in the total number of particles.

2.1 Considered gedanken experiment

The experimental procedure we consider to measure the phase coherence is inspired by recent experiments at JILA [9]. A condensate is first created in a trap in some internal atomic state a; the corresponding condensate wavefunction in the zero temperature mean-field approximation is ϕ_0, a stationary solution of the Gross-Pitaevskii equation:

$$\mu\phi_0 = -\frac{\hbar^2}{2m}\Delta\phi_0 + [U_{\rm a}(\mathbf{r}) + Ng_{\rm aa}|\phi_0|^2]\phi_0. \tag{1}$$

In this equation N is the number of particles, $g_{\rm aa}$ is the coupling constant between the atoms in the internal state a, related to the scattering length $a_{\rm aa}$ by $g_{\rm aa} = 4\pi\hbar^2 a_{\rm aa}/m$; $U_{\rm a}$ is the trapping potential seen by the atoms in a and μ is the chemical potential. Note that we have normalized ϕ_0 to unity.

At time $t = 0$ a resonant electromagnetic pulse transfers in a coherent way part of the atoms to a second internal state b. The state of the system is then given in the Hartree-Fock approximation by

$$|\Psi(0)\rangle = [c_{\rm a}|a,\phi_0\rangle + c_{\rm b}|b,\phi_0\rangle]^N \tag{2}$$

with $|c_{\rm a}|^2 + |c_{\rm b}|^2 = 1$. As we assume a Rabi coupling between a and b much more intense than $\mu/\hbar$ the atomic interactions have a negligible effect during the transfer so that the amplitudes $c_{\rm a,b}$ depend only on the pulse parameters, not on the number N of particles. In the N-particle state equation (2) the condensate in state a and the condensate in state b have a well-defined relative phase; we therefore call this state *a phase state*, in analogy with [13].

The two condensates evolve freely in their trapping potentials during the time τ. During this evolution we assume that there is no coherent coupling between a and b to lock the relative phase of the condensates; in particular the only considered interactions between the particles are elastic, of the type $a+a \to a+a$ (coupling constant $g_{\rm aa} > 0$), $a+b \to a+b$ (coupling constant $g_{\rm ab} > 0$), $b+b \to b+b$ (coupling constant $g_{\rm bb} > 0$). We therefore expect a collapse of the relative phase for sufficiently long times, due to atomic interactions.

To test the phase coherence at time τ, a second electromagnetic pulse is applied to mix the internal states a and b. We assume that this second pulse is a $\pi/2$ pulse, so that the atomic field operators in the Heisenberg picture are transformed according to

$$\hat{\psi}_{\rm a}(\tau^+) = \frac{{\rm e}^{-{\rm i}\delta}}{\sqrt{2}}\hat{\psi}_{\rm a}(\tau^-) + \frac{{\rm e}^{{\rm i}\delta}}{\sqrt{2}}\hat{\psi}_{\rm b}(\tau^-), \tag{3}$$

$$\hat{\psi}_{\rm b}(\tau^+) = -\frac{{\rm e}^{-{\rm i}\delta}}{\sqrt{2}}\hat{\psi}_{\rm a}(\tau^-) + \frac{{\rm e}^{{\rm i}\delta}}{\sqrt{2}}\hat{\psi}_{\rm b}(\tau^-), \tag{4}$$

δ being an adjustable phase. One then measures the mean spatial density $\rho_{\rm a}$ in the internal state a, averaging over many realizations of the whole experiment:

$$\rho_{\rm a} = \langle\hat{\psi}_{\rm a}^\dagger(\tau^+)\hat{\psi}_{\rm a}(\tau^+)\rangle. \tag{5}$$

The signature of a phase coherence between the two condensates at time τ is the dependence of the mean density $\rho_{\rm a}$ on the adjustable phase δ. More precisely we define the contrast

$$C = \frac{\max_\delta\rho_{\rm a} - \min_\delta\rho_{\rm a}}{\max_\delta\rho_{\rm a} + \min_\delta\rho_{\rm a}} = \frac{2|\langle\hat{\psi}_{\rm b}^\dagger(\tau^-)\hat{\psi}_{\rm a}(\tau^-)\rangle|}{\sum_{\varepsilon={\rm a,b}}\langle\hat{\psi}_\varepsilon^\dagger(\tau^-)\hat{\psi}_\varepsilon(\tau^-)\rangle}\cdot \tag{6}$$

The contrast involves the interference term $\langle\hat{\psi}_{\rm b}^\dagger(\tau^-)\hat{\psi}_{\rm a}(\tau^-)\rangle$ which carries the information about the relative phase between the two condensates.

2.2 Approximate evolution of an initial phase state

The time evolution in the phase state representation is not simple, as an initial phase state is mapped onto a superposition of phase states. It is more convenient to introduce Fock states, that is states with a well-defined number of particles in a and in b, these numbers being preserved by the time evolution.

We therefore expand the initial phase state over the Fock states:

$$|\Psi(0)\rangle = \sum_{N_\mathrm{a}=0}^{N} \left(\frac{N!}{N_\mathrm{a}!N_\mathrm{b}!}\right)^{1/2} c_\mathrm{a}^{N_\mathrm{a}} c_\mathrm{b}^{N_\mathrm{b}} |N_\mathrm{a}:\phi_0, N_\mathrm{b}:\phi_0\rangle \quad (7)$$

where we set $N_\mathrm{b} = N - N_\mathrm{a}$.

By calculating the evolution of each Fock state in the simplest Hartree-Fock approximation, we get the following mapping:

$$|N_\mathrm{a}:\phi_0, N_\mathrm{b}:\phi_0\rangle \rightarrow \mathrm{e}^{-\mathrm{i}A(N_\mathrm{a},N_\mathrm{b};t)/\hbar}|N_\mathrm{a}:\phi_\mathrm{a}(N_\mathrm{a},N_\mathrm{b};t), N_\mathrm{b}:\phi_\mathrm{b}(N_\mathrm{a},N_\mathrm{b};t)\rangle \quad (8)$$

where the condensates wavefunctions evolve according to the coupled Gross-Pitaevskii equations:

$$\mathrm{i}\hbar\partial_t\phi_\varepsilon = \Big[-\frac{\hbar^2}{2m}\Delta + U_\varepsilon(\mathbf{r}) + N_\varepsilon g_{\varepsilon\varepsilon}|\phi_\varepsilon|^2 + N_{\varepsilon'}g_{\varepsilon\varepsilon'}|\phi_{\varepsilon'}|^2\Big]\phi_\varepsilon \quad (9)$$

(where $\varepsilon' \neq \varepsilon$) with the initial conditions

$$\phi_\mathrm{a}(0) = \phi_\mathrm{b}(0) = \phi_0 \quad (10)$$

and where the time dependent phase factor A solves:

$$\frac{\mathrm{d}}{\mathrm{d}t}A(N_\mathrm{a},N_\mathrm{b};t) = -\frac{1}{2}N_\mathrm{a}^2 g_\mathrm{aa}\int \mathrm{d}\mathbf{r}\ |\phi_\mathrm{a}|^4 - \frac{1}{2}N_\mathrm{b}^2 g_\mathrm{bb}\int \mathrm{d}\mathbf{r}\ |\phi_\mathrm{b}|^4 - N_\mathrm{a}N_\mathrm{b}g_\mathrm{ab}\int \mathrm{d}\mathbf{r}\ |\phi_\mathrm{a}|^2|\phi_\mathrm{b}|^2. \quad (11)$$

Equation (11) is derived in Appendix A. Physically $\mathrm{d}A/\mathrm{d}t$ is simply the opposite of the mean interaction energy between the particles in the Fock state. In the case where the Fock state is a steady state, the need for the phase factor A additional to the Gross-Pitaevskii equation is obvious; the exact phase factor is indeed $\mathrm{e}^{-\mathrm{i}Et/\hbar}$, where E is the energy of the Fock state, whereas the phase factor obtained from the Gross-Pitaevskii evolution is $\mathrm{e}^{-\mathrm{i}(N_\mathrm{a}\mu_\mathrm{a}+N_\mathrm{b}\mu_\mathrm{b})t/\hbar}$, where $\mu_\mathrm{a,b}$ is the chemical potential in a, b.

Using the evolution of the Fock states, and other approximations valid in the limit of large numbers of particles (as detailed in the Appendix B) we obtain for the interference term between the condensates with a well-defined total number N of particles:

$$\langle\hat{\psi}_\mathrm{b}^\dagger\hat{\psi}_\mathrm{a}\rangle_N = c_\mathrm{a}c_\mathrm{b}^* \sum_{N_\mathrm{a}=1}^{N} \frac{N!}{(N_\mathrm{a}-1)!N_\mathrm{b}!}|c_\mathrm{a}|^{2(N_\mathrm{a}-1)}|c_\mathrm{b}|^{2N_\mathrm{b}} \times \phi_\mathrm{a}(N_\mathrm{a},N_\mathrm{b})\phi_\mathrm{b}^*(N_\mathrm{a}-1,N_\mathrm{b}+1) \quad (12)$$

where $N_\mathrm{b} = N - N_\mathrm{a}$. The exact computation of this sum remains a formidable task, since it involves in principle the solution of N different sets of two coupled Gross-Pitaevskii equations. We introduce some simplifying approximations in the next subsection.

2.3 Phase collapse for a mixture

In the present experiments the total number of particles fluctuates from one realization to the other, so that equation (12) has to be averaged over N. We assume that the fluctuations of the total number of particles have a standard deviation ΔN much smaller than the mean total particle number $\bar{N}$. As the distributions of the number of particles in a and b in a phase state have also a width much smaller than $\bar{N}$ (typically on the order of $\bar{N}^{1/2}$) we can assume than the number of particles in a and in b are very close to their average values $\bar{N}_\varepsilon = |c_\varepsilon|^2\bar{N}$. We now take advantage of this property to simplify equation (12).

We split the condensate wavefunctions in a modulus and a phase θ_ε; we assume that the variation of the modulus can be neglected over the distribution of $N_\mathrm{a,b}$, and that the variation of the phase can be approximated by a linear expansion around $\bar{N}_\varepsilon$. We thus get the approximate form for the condensate wavefunctions:

$$\phi_\varepsilon(N_\mathrm{a},N_\mathrm{b}) \simeq \bar{\phi}_\varepsilon \exp\left[\mathrm{i}\sum_{\varepsilon'=\mathrm{a,b}}(N_{\varepsilon'}-\bar{N}_{\varepsilon'})(\partial_{N_{\varepsilon'}}\theta_\varepsilon)(\bar{N}_\mathrm{a},\bar{N}_b)\right] \quad (13)$$

where $\bar{\phi}_\varepsilon = \phi_\varepsilon(N_\mathrm{a} = \bar{N}_\mathrm{a}, N_\mathrm{b} = \bar{N}_\mathrm{b})$.

To this level of approximation the mean densities in the internal states a, b are simply given by

$$\langle\hat{\psi}_\varepsilon^\dagger\hat{\psi}_\varepsilon\rangle_N \simeq \bar{N}_\varepsilon|\bar{\phi}_\varepsilon|^2 \quad (14)$$

whereas the interference term between the condensates is:

$$\langle\hat{\psi}_\mathrm{b}^\dagger\hat{\psi}_\mathrm{a}\rangle_N \simeq \bar{N}c_\mathrm{a}c_\mathrm{b}^*\bar{\phi}_\mathrm{a}\bar{\phi}_\mathrm{b}^*\exp\left\{\mathrm{i}[(N-\bar{N})\chi_\mathrm{s} - \bar{N}(|c_\mathrm{a}|^2-|c_\mathrm{b}|^2)\chi_\mathrm{d}]\right\}\mathrm{e}^{\mathrm{i}\chi_0} \times \left[|c_\mathrm{a}|^2\mathrm{e}^{\mathrm{i}\chi_\mathrm{d}} + |c_\mathrm{b}|^2\mathrm{e}^{-\mathrm{i}\chi_\mathrm{d}}\right]^{N-1}. \quad (15)$$

In this last expression we have introduced the time and position dependent quantities

$$\chi_\mathrm{s} = \frac{1}{2}\left[(\partial_{N_\mathrm{a}} + \partial_{N_\mathrm{b}})(\theta_\mathrm{a}-\theta_\mathrm{b})\right](\bar{N}_\mathrm{a},\bar{N}_\mathrm{b}) \quad (16)$$

$$\chi_\mathrm{d} = \frac{1}{2}\left[(\partial_{N_\mathrm{a}} - \partial_{N_\mathrm{b}})(\theta_\mathrm{a}-\theta_\mathrm{b})\right](\bar{N}_\mathrm{a},\bar{N}_\mathrm{b}). \quad (17)$$

The phase $\chi_0 = (1/2)(\partial_{N_\mathrm{a}} - \partial_{N_\mathrm{b}})(\theta_\mathrm{a}+\theta_\mathrm{b})(\bar{N}_\mathrm{a},\bar{N}_\mathrm{b})$ in equation (15) is less important as contrarily to $\chi_\mathrm{s,d}$ it is not multiplied by N. At time $t = 0$ all the χ's vanish. In the large N limit, the χ's are expected to be on the order of $\bar{\mu}t/\hbar\bar{N}$.

The factor responsible for the collapse of the contrast at a fixed value of N is the last line of equation (15), the exponential factors in the first two lines being of modulus one. As N is large a small variation of χ_d from its initial value $\chi_\mathrm{d}(t=0) = 0$ is sufficient to destroy the interference term. Over the range of the collapse we can therefore expand the exponential of $\pm\mathrm{i}\chi_\mathrm{d}$ to second order in χ_d, obtaining:

$$\langle\hat{\psi}_\mathrm{b}^\dagger\hat{\psi}_\mathrm{a}\rangle_N \simeq \bar{N}c_\mathrm{a}c_\mathrm{b}^*\bar{\phi}_\mathrm{a}\bar{\phi}_\mathrm{b}^*\exp\left\{\mathrm{i}(N-\bar{N}) \times [\chi_\mathrm{s} + (|c_\mathrm{a}|^2-|c_\mathrm{b}|^2)\chi_\mathrm{d}]\right\}\ \exp\left[-2N\chi_\mathrm{d}^2|c_\mathrm{a}|^2|c_\mathrm{b}|^2\right]. \quad (18)$$

The second exponential factor in this expression allows to determine the collapse time $t_\mathrm{c}^\mathrm{fix}$ for a fixed number of particles, through the condition

$$4N|c_\mathrm{a}|^2|c_\mathrm{b}|^2\chi_\mathrm{d}^2(t_\mathrm{c}^\mathrm{fix}) \simeq 1 \quad (19)$$

such that the modulus of the interference term is reduced by a factor $e^{-1/2}$ from its initial value. The first exponential factor in equation (18) accounts for the phase difference of the interference term for N particles and $\bar{N}$ particles, as shown by the identity:

$$\chi_{\rm s} + (|c_{\rm a}|^2 - |c_{\rm b}|^2)\chi_{\rm d} = \frac{\rm d}{{\rm d}N}\left[(\theta_{\rm a} - \theta_{\rm b})(N|c_{\rm a}|^2, N|c_{\rm b}|^2)\right]_{N=\bar{N}}. \quad (20)$$

This phase factor can also be understood as a consequence of a drift of the relative phase between two condensates at a velocity $v(N)$ depending on the total number of particles:

$$v(N) = \partial_t(\bar{\theta_{\rm a}} - \bar{\theta_{\rm b}}) + (N - \bar{N})\partial_t\left[\chi_{\rm s} + (|c_{\rm a}|^2 - |c_{\rm b}|^2)\chi_{\rm d}\right]. \quad (21)$$

As we shall see in the next subsection fluctuations in the total number of particles N result in fluctuations of this phase factor, providing an additional source of smearing of the phase, as already emphasized in [8].

2.4 Effect of fluctuations in the total number of particles

The effect on the phase collapse of fluctuations in the total number of particles is obtained by averaging equation (18) over the probability distribution of N. To be specific we assume a Gaussian distribution for N. The average can be calculated by replacing the discrete sum over N by an integral; we neglect a term proportional to $(\Delta N\chi_{\rm d}^2)^2$ scaling as $(\Delta N/\bar{N})^2$ at the collapse time $t_{\rm c}^{\rm fix}$; the resulting modulus of the interference term reads:

$$|\langle\hat{\psi}_{\rm b}^{\dagger}\hat{\psi}_{\rm a}\rangle^{\rm Gauss}| \simeq \bar{N}|c_{\rm a}c_{\rm b}^{*}\bar{\phi_{\rm a}}\bar{\phi_{\rm b}}^{*}|\exp\left\{-\frac{1}{2}(\Delta N)^2 \times\left[\frac{\rm d}{{\rm d}N}(\theta_{\rm a} - \theta_{\rm b})\right]^2_{N=\bar{N}}\right\}\exp\left[-2\bar{N}\chi_{\rm d}^2|c_{\rm a}|^2|c_{\rm b}|^2\right]. \quad (22)$$

The first exponential factor in this expression represents the damping of the interference term due to fluctuations in the total number of particles; the second exponential factor, already present in equation (18), gives the damping due to fluctuations in the relative number of particles between a and b, as can be seen in equation (17).

2.5 The steady state case and comparison with previous treatments

Our treatment can be easily adapted to the case of two initially different condensate wavefunctions $\phi_{\rm a}(t=0)$ and $\phi_{\rm b}(t=0)$. In the particular case of condensates in stationary states, the formulas for the interference term $\langle\hat{\psi_{\rm b}}^{\dagger}\hat{\psi_{\rm a}}\rangle$ remain the same, and one has $\theta_\varepsilon = -\mu_\varepsilon(N_{\rm a}, N_{\rm b})t/\hbar$. We can give in this case the explicit expression for the collapse time $t_{\rm c}^{\rm fix}$ defined in equation (19), assuming a fixed total number of atoms $N = \bar{N}$:

$$t_{\rm c}^{\rm fix} = \hbar\left[\bar{N}^{1/2}|c_{\rm a}c_{\rm b}||(\partial_{N_{\rm a}} - \partial_{N_{\rm b}})(\mu_{\rm a} - \mu_{\rm b})|\right]^{-1}. \quad (23)$$

For the particular case of non mutually interacting steady state condensates μ_ε depends on N_ε only, so that the partial derivatives in the denominator of equation (23) reduce to ${\rm d}\mu_{\rm a}/{\rm d}N_{\rm a} + {\rm d}\mu_{\rm b}/{\rm d}N_{\rm b}$, and we recover the results of [5,8].

From equation (23) we see that what matters physically is the change in the *difference* between the chemical potentials of the two condensates when one transfers one particle from one condensate to the other. For this reason the case of mutually interacting condensates with close coupling constants can lead to much larger $t_{\rm c}$'s as compared to the case of non-mutually interacting condensates. For example, in the case of the JILA experiment [9], assuming that the condensates are in steady state, one finds $t_{\rm c}^{\rm fix} \simeq 3.1$ s; by ignoring the interaction between the condensates (setting by hand $g_{\rm ab} = 0$) one obtains the much shorter time $\simeq 0.25$ s. The JILA case is analyzed in more detail in our section 5.

A similar prediction on the reduction of decoherence due to mutual interactions between the two condensates, in trapping potentials with different curvatures, was obtained numerically in [11].

The treatment in [7] considers the absolute phase dynamics of a single condensate (in our formalism $c_{\rm b} = 0$) in a coherent state. When the condensate wavefunction is stationary one has simply $\theta_{\rm a} = -\mu_{\rm a}t/\hbar$. From equation (22) with $\Delta N = \bar{N}^{1/2}$ (as the coherent state has a Poisson distribution for N) we then find that the phase of the condensate order parameter is damped as $\exp[-\bar{N}({\rm d}\mu_{\rm a}/{\rm d}N)^2t^2/2\hbar^2]$ as in [7].

3 Application to the breathe-together solution

In this section we consider a particular solution of the coupled Gross-Pitaevskii equations for which an approximate scaling solution is available when the chemical potential is much larger than the energy spacing between trap levels, the so-called Thomas-Fermi regime. We first give the set of parameters for which this solution, that we call the breathe-together solution, exists. We then linearize the Gross-Pitaevskii equations around this solution to determine its stability with respect to demixing and to obtain the phase coherence dynamics.

3.1 Description of the breathe-together solution

We now determine the set of parameters such that the coupled Gross-Pitaevskii equations (Eq. (9)) for

$$N_\varepsilon = \bar{N}_\varepsilon \equiv \bar{N}|c_\varepsilon|^2 \quad (24)$$

have a solution with $\bar\phi_{\rm a}(\mathbf{r},t) = \bar\phi_{\rm b}(\mathbf{r},t) \equiv \bar\phi(\mathbf{r},t)$. The general condition is that the effective potential, that is the trapping potential plus the mean field potential, seen by the atoms in a and in b should be the same. This condition is satisfied when:

$$U_{\rm a}(\mathbf{r}) = U_b(\mathbf{r}) \equiv U(\mathbf{r}) \tag{25}$$

$$\bar N_{\rm a} g_{\rm aa} + \bar N_{\rm b} g_{\rm ab} = \bar N_{\rm b} g_{\rm bb} + \bar N_{\rm a} g_{\rm ab} \equiv \bar N g. \tag{26}$$

The resulting Gross-Pitaevskii equation for the condensate wavefunction $\bar\phi$ common to a and b is then:

$$\mathrm{i}\hbar\partial_t\bar\phi = \left[-\frac{\hbar^2}{2m}\mathit{\Delta} + U(\mathbf{r}) + \bar N g|\bar\phi|^2\right]\bar\phi \tag{27}$$

with the initial condition $\bar\phi(\mathbf{r},0) = \phi_0[N=\bar N](\mathbf{r}) \equiv \bar\phi_0$, where ϕ_0 is defined in equation (1).

By rewriting equation (26) as $\bar N_{\rm a}/\bar N_{\rm b} = (g_{\rm bb} - g_{\rm ab})/(g_{\rm aa} - g_{\rm ab})$ we see that this equality can be satisfied by choosing properly the mixing angle between a and b provided that

$$g_{\rm ab} < g_{\rm aa}, g_{\rm bb} \quad \text{or} \quad g_{\rm ab} > g_{\rm aa}, g_{\rm bb}. \tag{28}$$

As we shall see below, only the first case is relevant here, since the second case corresponds to an unstable solution with respect to demixing between a and b.

3.2 Linearization around the breathe-together solution

The strategy to obtain the quantities $\chi_{\rm s,d}$ relevant for the phase dynamics is to calculate in the linear approximation the deviations $\delta\phi_\varepsilon$ between the breathe-together solution $\bar\phi$ and neighboring solutions ϕ_ε for N_ε slightly different from $\bar N_\varepsilon$:

$$\delta\phi_\varepsilon \equiv \phi_\varepsilon(\bar N_{\rm a} + \delta N_{\rm a}, \bar N_{\rm b} + \delta N_{\rm b}) - \phi_\varepsilon(\bar N_{\rm a}, \bar N_{\rm b}). \tag{29}$$

From the definitions equations (16, 17) one indeed realizes that in the limit of small $\delta N_{\rm a}$:

$$\chi_{\rm s} = \left[\frac{\delta\theta_{\rm a} - \delta\theta_{\rm b}}{2\delta N_{\rm a}}\right]_{\delta N_{\rm b} = \delta N_{\rm a}} \tag{30}$$

$$\chi_{\rm d} = \left[\frac{\delta\theta_{\rm a} - \delta\theta_{\rm b}}{2\delta N_{\rm a}}\right]_{\delta N_{\rm b} = -\delta N_{\rm a}} \tag{31}$$

where $\delta\theta_{\rm a,b}$ are the deviations of the phase of the neighboring solutions ϕ_ε from the phase of the breathe-together solution:

$$\delta\theta_{\rm a} - \delta\theta_{\rm b} = \mathrm{Im}\left[\frac{\delta\phi_{\rm a}}{\bar\phi} - \frac{\delta\phi_{\rm b}}{\bar\phi}\right]. \tag{32}$$

It turns out that homogeneous rather than inhomogeneous linear equations can be obtained for the deviations $\delta\phi_\epsilon$ by introducing the quantities:

$$\delta\varphi_\varepsilon \equiv \frac{\delta[\sqrt{N_\varepsilon}\phi_\varepsilon]}{\sqrt{\bar N_\varepsilon}} = \frac{\delta N_\varepsilon}{2\bar N_\varepsilon}\bar\phi + \delta\phi_\varepsilon. \tag{33}$$

Furthermore a partial decoupling occurs for the linear combinations

$$\delta\varphi_{\rm s} \equiv \delta\varphi_{\rm a} + \delta\varphi_{\rm b} \tag{34}$$

$$\delta\varphi_{\rm d} \equiv \delta\varphi_{\rm a} - \delta\varphi_{\rm b}. \tag{35}$$

The sum $\delta\varphi_{\rm s}$ obeys a linear equation involving $\delta\varphi_{\rm d}$ as a source term:

$$\begin{aligned}\mathrm{i}\hbar\partial_t\delta\varphi_{\rm s} = &\left[-\frac{\hbar^2}{2m}\mathit{\Delta} + U + 2\bar N g|\bar\phi|^2\right]\delta\varphi_{\rm s}\\ &+ \bar N g\bar\phi^2\delta\varphi_{\rm s}^* + (\bar N_{\rm a} g_{\rm aa} - \bar N_{\rm b} g_{\rm bb})(|\bar\phi|^2\delta\varphi_{\rm d} + \bar\phi^2\delta\varphi_{\rm d}^*).\end{aligned} \tag{36}$$

The part of this equation involving $\delta\varphi_{\rm s}$ is identical to the one obtained for a single condensate with $\bar N$ particles and a coupling constant g. The corresponding modes have minimal frequencies on the order of the trap frequency ω for an isotropic harmonic trap [14].

The difference $\delta\varphi_{\rm d}$ obeys the closed equation:

$$\begin{aligned}\mathrm{i}\hbar\partial_t\delta\varphi_{\rm d} = &\left[-\frac{\hbar^2}{2m}\mathit{\Delta} + U + \bar N g|\bar\phi|^2\right]\delta\varphi_{\rm d}\\ &+ \frac{\bar N_{\rm a}\bar N_{\rm b}}{\bar N}(g_{\rm aa} + g_{\rm bb} - 2g_{\rm ab})(|\bar\phi|^2\delta\varphi_{\rm d} + \bar\phi^2\delta\varphi_{\rm d}^*)\end{aligned} \tag{37}$$

where we have used the identity:

$$\begin{aligned}\bar N_{\rm b}(g_{\rm bb} - g_{\rm ab}) &= \bar N_{\rm a}(g_{\rm aa} - g_{\rm ab})\\ &= \frac{\bar N_{\rm a}\bar N_{\rm b}}{\bar N}(g_{\rm aa} + g_{\rm bb} - 2g_{\rm ab}).\end{aligned} \tag{38}$$

As shown in [15] minimal eigenfrequencies of this equation can be much smaller than ω; *e.g.* when all the coupling constants are equal, the minimal eigenfrequencies in a harmonic isotropic trap of frequency ω scale as $\hbar\omega^2/\mu \ll \omega$ in the Thomas-Fermi limit.

For the derivation of the χ's it is sufficient to calculate $\delta\varphi_{\rm d}$. The relative phase between the two condensates for the considered neighboring solution with $N_\varepsilon = \bar N_\varepsilon + \delta N_\varepsilon$ particles in the state ε is in fact given by:

$$\delta\theta_{\rm a} - \delta\theta_{\rm b} = \frac{1}{2\mathrm{i}}\left[\frac{\delta\varphi_{\rm d}}{\bar\phi} - \frac{\delta\varphi_{\rm d}^*}{\bar\phi^*}\right]. \tag{39}$$

as can be checked from the definition equation (32).

3.3 Approximate equations of evolution in the Thomas-Fermi limit

In the remaining part of this section we assume an isotropic harmonic trapping potential $U(\mathbf{r}) = m\omega^2 r^2/2$ and we restrict to the Thomas-Fermi limit $\mu \gg \hbar\omega$.

In the Thomas-Fermi limit it is known [16,17] that most of the time dependence of the wavefunction $\bar\phi$ can be absorbed by a time dependent gauge and scaling transform; here we apply this transform to both $\bar\phi$ and $\delta\varphi_{\rm d}$:

$$\bar\phi(\mathbf{r},t) \equiv \frac{\mathrm{e}^{-\mathrm{i}\eta(t)}}{\lambda^{3/2}(t)}\mathrm{e}^{imr^2\dot\lambda(t)/2\hbar\lambda(t)}\tilde{\bar\phi}(\mathbf{r}/\lambda(t),t) \tag{40}$$

$$\delta\varphi_{\rm d}(\mathbf{r},t) = \frac{\mathrm{e}^{-\mathrm{i}\eta(t)}}{\lambda^{3/2}(t)}\mathrm{e}^{imr^2\dot\lambda(t)/2\hbar\lambda(t)}\delta\tilde\varphi_{\rm d}(\mathbf{r}/\lambda(t),t). \tag{41}$$

The scaling factor $\lambda(t)$ solves the second order Newton-type differential equation

$$\ddot{\lambda} = \frac{g}{g_{\mathrm{aa}}} \frac{\omega^2}{\lambda^4} - \omega^2 \lambda \tag{42}$$

with the initial condition $\lambda(0) = 1, \dot{\lambda}(0) = 0$. The "force" seen by λ in equation (42) derives from the sum of an expelling $1/\lambda^3$ potential due to repulsive interactions between atoms and an attractive λ^2 potential due to the harmonic confinement of the atoms. It leads to periodic oscillations of λ, that is to a periodic breathing of the condensates. We have also introduced a phase factor involving the time dependent function η such that $\dot{\eta} = \bar{\mu} g/(g_{\mathrm{aa}} \lambda^3 \hbar)$.

In the Appendix C we derive approximate evolution equations for $\tilde{\bar{\phi}}$ and $\delta\tilde{\varphi}_{\mathrm{d}}$; we give here only the result. To lowest order in the Thomas-Fermi approximation $\tilde{\bar{\phi}}$ does not evolve and can be approximated by the Thomas-Fermi approximation for $\bar{\phi}_0$:

$$\tilde{\bar{\phi}}(\mathbf{r}, t) \simeq \bar{\phi}_0(\mathbf{r}) \simeq \left(\frac{15}{8\pi R_0^3} \right)^{1/2} \left[1 - \frac{r^2}{R_0^2} \right]^{1/2} \tag{43}$$

with a Thomas-Fermi radius $R_0 = \sqrt{2\bar{\mu}/m\omega^2}$. The approximate evolution for $\delta\tilde{\varphi}_{\mathrm{d}}$ is conveniently expressed in terms of the real function α and the purely imaginary function β:

$$\alpha = \tilde{\bar{\phi}}^* \delta\tilde{\varphi}_{\mathrm{d}} + \tilde{\bar{\phi}} \delta\tilde{\varphi}_{\mathrm{d}}^* \tag{44}$$

$$\beta = \frac{1}{2} \left[\frac{\delta\tilde{\varphi}_{\mathrm{d}}}{\tilde{\bar{\phi}}} - \frac{\delta\tilde{\varphi}_{\mathrm{d}}^*}{\tilde{\bar{\phi}}^*} \right]. \tag{45}$$

These functions have a clear physical meaning. The first one α corresponds to the deviation $\delta\rho_{\mathrm{a}}/\bar{N}_{\mathrm{a}} - \delta\rho_{\mathrm{b}}/\bar{N}_{\mathrm{b}}$ written in the rescaled frame, $\delta\rho_\varepsilon$ being the deviation of spatial density in the condensate ε from the breathe-together solution. Apart from a factor i the second function β is the deviation of the relative phase equation (39) written in the rescaled frame:

$$(\delta\theta_{\mathrm{a}} - \delta\theta_{\mathrm{b}})(\mathbf{r}, t) = -\mathrm{i}\beta(\mathbf{r}/\lambda, t). \tag{46}$$

The equations of evolution for α, β are:

$$\mathrm{i}\hbar\partial_t \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = L(t) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{47}$$

where the operator $L(t)$ in the Thomas-Fermi approximation reads:

$$L(t) = \begin{pmatrix} 0 & -\frac{\hbar^2}{m\lambda^2} \, \mathrm{div} \left[\bar{\phi}_0^{\,2} \, \mathbf{grad}(\cdot) \right] \\ \frac{1}{\lambda^3} \frac{\bar{N}_{\mathrm{a}} \bar{N}_{\mathrm{b}}}{\bar{N}} (g_{\mathrm{aa}} + g_{\mathrm{bb}} - 2g_{\mathrm{ab}}) & 0 \end{pmatrix}. \tag{48}$$

The initial conditions for α, β at time $t = 0$ obtained from equations (33, 10) are:

$$\alpha(0) = \left(\frac{\delta N_{\mathrm{a}}}{\bar{N}_{\mathrm{a}}} - \frac{\delta N_{\mathrm{b}}}{\bar{N}_{\mathrm{b}}} \right) \bar{\phi}_0^{\,2} \tag{49}$$

$$\beta(0) = 0. \tag{50}$$

3.4 Solution of the Thomas-Fermi evolution equations: stability against demixing

The strategy to determine the time evolution of α, β is (1) to expand the vector $(\alpha(0), \beta(0))$ on eigenmodes of the operator $L(0)$, and (2) to calculate the time evolution of each eigenmode.

3.4.1 Expansion on modes of L(0)

Consider an eigenvector (α, β) of the operator $L(0)$ with the eigenvalue $\hbar\Omega$. For $\Omega \neq 0$ one can express the component β as function of α:

$$\beta = \frac{\alpha}{\hbar\Omega} \frac{\bar{N}_{\mathrm{a}} \bar{N}_{\mathrm{b}}}{\bar{N}} (g_{\mathrm{aa}} + g_{\mathrm{bb}} - 2g_{\mathrm{ab}}) \tag{51}$$

and obtain the eigenvalue problem for α:

$$\Omega^2 \alpha = \left(\frac{\bar{N}_{\mathrm{a}} \bar{N}_{\mathrm{b}}}{\bar{N}^2} \frac{(g_{\mathrm{aa}} + g_{\mathrm{bb}} - 2g_{\mathrm{ab}})}{g_{\mathrm{aa}}} \right) S[\alpha] \tag{52}$$

where we have introduced the Stringari operator:

$$S[\alpha] \equiv -\frac{\bar{N} g_{\mathrm{aa}}}{m} \, \mathrm{div} [\bar{\phi}_0^{\,2} \, \mathbf{grad} \, \alpha]. \tag{53}$$

This operator has been studied in [14]. It is an Hermitian and positive operator, with a spectrum $q\omega^2$, q non negative integer; q is given by

$$q = 2n^2 + 2nl + 3n + l \tag{54}$$

as function of the radial quantum number n and the angular momentum l. This allows the determination of the eigenfrequencies Ω:

$$\Omega_q = \pm \left(\frac{\bar{N}_{\mathrm{a}} \bar{N}_{\mathrm{b}}}{\bar{N}^2} \frac{(g_{\mathrm{aa}} + g_{\mathrm{bb}} - 2g_{\mathrm{ab}})}{g_{\mathrm{aa}}} \right)^{1/2} q^{1/2} \omega, \tag{55}$$

with $q > 0$ as we have assumed $\Omega \neq 0$. The case of a vanishing Ω corresponds to the zero energy mode $\alpha_0 = 0, \beta_0 = 1$ of the operator $L(0)$, as it can be checked from a direct calculation.

All the eigenmodes of $L(0)$ have been identified. They do not form a complete family of vectors however. The vector $(\alpha = 1, \beta = 0)$ cannot be expanded on the eigenmodes of $L(0)$. Its first component α is indeed in the kernel of the operator S (as $S[\alpha] = 0$) whereas none of the α_q is in the kernel of S ($S[\alpha_q] = q\omega^2 \alpha_q$ is not identically zero) except when α_q is identically zero (for $q = 0$). The family of eigenvectors of $L(0)$ completed by the additional vector $(\alpha = 1, \beta = 0)$ forms a basis. The additional vector is called an anomalous mode, and we set $\alpha_{\mathrm{anom}} = 1$, $\beta_{\mathrm{anom}} = 0$; the action of $L(0)$ on the anomalous mode gives the zero energy mode times the constant factor $\bar{N}_{\mathrm{a}}(g_{\mathrm{aa}} - g_{\mathrm{ab}})$ [18].

The mode functions α_q of the operator S are given in [14]. It turns out that in the expansion of the initial conditions for α, β equations (49, 50), only the isotropic eigenmodes of $L(0)$ with $q = 5$ and the anomalous mode are involved:

$$\begin{pmatrix}\alpha(0)\\ \beta(0)\end{pmatrix} = C_5 \left[\begin{pmatrix}\alpha_{q=5}\\ \beta_{q=5}\end{pmatrix} + \begin{pmatrix}\alpha_{q=5}\\ -\beta_{q=5}\end{pmatrix}\right] + C_{\mathrm{anom}}\begin{pmatrix}1\\ 0\end{pmatrix}. \tag{56}$$

The isotropic eigenmode of S with $q = 5$, the so-called breathing mode, reads

$$\alpha_{q=5}(\mathbf{r}) = \left[\frac{r^2}{R_0^2} - \frac{3}{5}\right]. \tag{57}$$

By equation (51) we have $\beta_{q=5} = \alpha_{q=5}\bar{N}_{\mathrm{a}}\bar{N}_{\mathrm{b}}(g_{\mathrm{aa}} + g_{\mathrm{bb}} - 2g_{\mathrm{ab}})/\bar{N}\hbar\Omega_{q=5}$. For the coefficients of the modal expansion of $(\alpha(0), \beta(0))$, we obtain

$$C_{\mathrm{anom}} = \frac{3}{4\pi R_0^3}\left(\frac{\delta N_{\mathrm{a}}}{\bar{N}_{\mathrm{a}}} - \frac{\delta N_{\mathrm{b}}}{\bar{N}_{\mathrm{b}}}\right) \tag{58}$$

$$C_5 = -\frac{5}{4}C_{\mathrm{anom}}. \tag{59}$$

3.4.2 Evolution of the modes and stability against demixing

As a second step we determine the time evolution of the modes of the operator $L(0)$. If we consider an eigenstate $(\alpha_q(0), \beta_q(0))$ of $L(0)$ with the eigenenergy $\hbar\Omega_q$ and evolve it according to equation (47), we find that the evolution reduces to multiplication by purely time dependent factors $A_q(t), B_q(t)$:

$$\alpha_q(\mathbf{r}, t) = A_q(t)\alpha(\mathbf{r}, 0) \tag{60}$$

$$\beta_q(\mathbf{r}, t) = B_q(t)\beta(\mathbf{r}, 0) \tag{61}$$

where the factors satisfy the differential equations:

$$\mathrm{i}\dot{A}_q = \frac{\Omega_q}{\lambda^2}B_q \tag{62}$$

$$\mathrm{i}\dot{B}_q = \frac{\Omega_q}{\lambda^3}A_q \tag{63}$$

with the initial conditions $A_q(0) = B_q(0) = 1$. Note that the zero energy eigenmode does not evolve, as $\Omega_q = 0$. The anomalous mode has to be integrated separately, leading to

$$\alpha_{\mathrm{anom}}(\mathbf{r}, t) = 1 \tag{64}$$

$$\beta_{\mathrm{anom}}(\mathbf{r}, t) = \frac{\bar{N}_{\mathrm{a}}\bar{N}_{\mathrm{b}}}{\bar{N}}\frac{(g_{\mathrm{aa}} + g_{\mathrm{bb}} - 2g_{\mathrm{ab}})}{\mathrm{i}\hbar}\int_0^t \frac{\mathrm{d}t'}{\lambda^3(t')}. \tag{65}$$

We are now able to address the problem of dynamical stability of the breathe-together solution. Dynamical stability requires that any small deviation of the ϕ_ε's from the breathe-together solution $\bar{\phi}$ should not grow exponentially with time. Here an exponential growth of α may correspond to a demixing of the two condensates a and b.

A first case of instability occurs when $g_{\mathrm{ab}} > g_{\mathrm{aa}}, g_{\mathrm{bb}}$. In this case the eigenfrequencies Ω_q are purely imaginary and A_q, B_q diverge exponentially with time [19]. We have checked by a numerical integration of the Gross-Pitaevskii equations with spherical symmetry that the spatial distribution then acquires a structure of alternating shells of a atoms and b atoms (see Fig. 1).

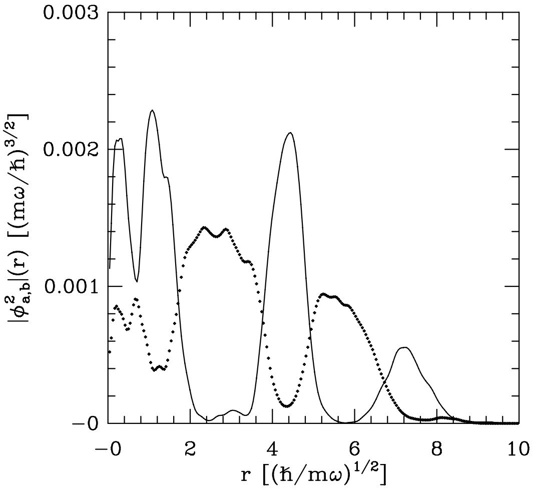

Fig. 1. Modulus squared of the condensate wavefunctions $|\phi^2_{\mathrm{a,b}}|(N_{\mathrm{a}}, N_{\mathrm{b}})$ as function of the distance r to the trap center at a time $\omega t \simeq 29.5$, from a numerical solution of the coupled Gross-Pitaevskii equations in the case of a dynamically unstable breathe-together solution. We have taken $g_{\mathrm{bb}}/g_{\mathrm{aa}} = 1.2$ and $g_{\mathrm{ab}}/g_{\mathrm{aa}} = 1.5$. We have applied a deviation $\delta N_{\mathrm{a}} = -\delta N_{\mathrm{b}} = -0.05\bar{N}_{\mathrm{a}}$ from the exact breathe-together condition. The chemical potential is $\bar{\mu} = 28.9\hbar\omega$. The curve in solid line corresponds to ϕ_{a}, the dotted curve corresponds to ϕ_{b}.

We suppose from now on that $g_{\mathrm{ab}} < g_{\mathrm{aa}}, g_{\mathrm{bb}}$. Instability may still occur in this case due to the periodic time dependence of the coefficients in the system (63), as shown in [20]. We have studied in more detail the stability of the mode $q = 5$, which is the one populated initially (see Eq. (56)); we have found non-zero instability exponents σ ($C_5(t) \sim \mathrm{e}^{\sigma t}$) in a very limited region of the plane $(g_{\mathrm{ab}}/g_{\mathrm{aa}}, g_{\mathrm{bb}}/g_{\mathrm{aa}})$, with very small exponents ($\sigma < 3 \times 10^{-2}\omega$). A direct numerical integration of the Gross-Pitaevskii equations did not show any demixing of a and b even at times $\gg \sigma^{-1}$ [21]. This suggests that the finite instability exponent is an artifact of the Thomas-Fermi approximation.

We assume in what follows the dynamical stability of the breathe-together solution.

3.5 Phase dynamics

In order to calculate the functions $\chi_{\mathrm{d}}, \chi_{\mathrm{s}}$ relevant for the relative phase dynamics, we calculate the evolution of the deviation $\delta\varphi_{\mathrm{d}}$ due to a small change in $N_{\mathrm{a}}, N_{\mathrm{b}}$ with respect to $\bar{N}_{\mathrm{a}}, \bar{N}_{\mathrm{b}}$, that is we evolve the initial state (56) according to the results of the previous subsection.

As we assume dynamical stability of the breathe-together solution, the modes with $q = 5$ perform only oscillations in time [22]. The relevant contribution for the phase dynamics therefore comes from the anomalous mode, which from equation (65) has a β diverging linearly with time. Assuming $\beta(\mathbf{r}, t) \sim C_{\mathrm{anom}}\beta_{\mathrm{anom}}(t)$

and using equations (58, 46) we obtain:

$$(\delta\theta_\mathrm{a}-\delta\theta_\mathrm{b})(\mathbf{r},t)\sim-\frac{2\bar{\mu}}{5}\frac{\bar{N}_\mathrm{a}\bar{N}_\mathrm{b}}{\bar{N}^2}\frac{(g_\mathrm{aa}+g_\mathrm{bb}-2g_\mathrm{ab})}{g_\mathrm{aa}}\times\left(\frac{\delta N_\mathrm{a}}{\bar{N}_\mathrm{a}}-\frac{\delta N_\mathrm{b}}{\bar{N}_\mathrm{b}}\right)\int_0^t\frac{\mathrm{d}t'}{\lambda^3(t')}\cdot \quad (66)$$

We specialize this formula with $\delta N_\mathrm{b}=\pm\delta N_\mathrm{a}$ and we get from equations (30, 31):

$$\chi_\mathrm{d}\sim-\frac{1}{2\hbar}\left(\frac{\mathrm{d}\mu}{\mathrm{d}N}\right)_{N=\bar{N}}\frac{g_\mathrm{aa}+g_\mathrm{bb}-2g_\mathrm{ab}}{g_\mathrm{aa}}\int_0^t\frac{\mathrm{d}t'}{\lambda^3(t')} \quad (67)$$

$$\chi_\mathrm{s}\sim(|c_\mathrm{b}|^2-|c_\mathrm{a}|^2)\chi_\mathrm{d}. \quad (68)$$

We have introduced the derivative of the chemical potential with respect to the total number of particles $((\mathrm{d}\mu/\mathrm{d}N)(N=\bar{N})\simeq2\bar{\mu}/5\bar{N}$ in the Thomas-Fermi limit) in order to recover the characteristic time scale for the phase collapse of steady state non mutually interacting condensates. Our formula reveals the interest of close coupling constants, such that $g_\mathrm{aa}+g_\mathrm{bb}-2g_\mathrm{ab}\ll g_\mathrm{aa}$. In this case χ_d is strongly reduced with respect to non mutually interacting condensates; λ performs small oscillations around the value $\lambda=1$ so that the integral over t' can be replaced by t. The more general case of close g's not necessarily satisfying the breathe together condition is analyzed in the next section.

We note that the value of χ_s as function of χ_d could be expected *a priori* from equation (20): when equation (26) is satisfied, the condensate wavefunctions form a breathe-together solution and have therefore a vanishing relative phase for $N_\mathrm{a}=N|c_\mathrm{a}|^2, N_\mathrm{b}=N|c_\mathrm{b}|^2$, whatever the value of N is. An important consequence is that there is no extra damping of the phase coherence due to the fluctuations of the total number of particles (see Eq. (22)).

4 Case of close coupling constants

We consider in this section the case of close coupling constants which leads to a dramatic reduction of the relative phase decoherence with respect to the case of non mutually interacting condensates.

The strategy is to solve approximately the Gross-Pitaevskii equations (Eq. (9)) for $\phi_\mathrm{a}(N_\mathrm{a},N_\mathrm{b})$ and $\phi_\mathrm{b}(N_\mathrm{a},N_\mathrm{b})$ and apply the formulas (16, 17) directly. For all equal g's the initial state is indeed a steady state for the equation (9) and $\chi_\mathrm{s}=\chi_\mathrm{d}=0$. For close g's we linearize the Gross-Pitaevskii equations around the initial value in the hydrodynamic point of view.

4.1 Linearization in the classical hydrodynamics approximation

We first rewrite the Gross-Pitaevskii equations (Eq. (9)) in terms of the hydrodynamic variables:

$$\rho_\varepsilon\equiv N_\varepsilon|\phi_\varepsilon(N_\mathrm{a},N_\mathrm{b})|^2 \quad (69)$$

$$\mathbf{v}_\varepsilon\equiv\frac{\hbar}{m}\,\mathbf{grad}\;\theta_\varepsilon(N_\mathrm{a},N_\mathrm{b}) \quad (70)$$

that is densities and velocity fields of the two condensates. We further assume the Thomas-Fermi limit $\mu\gg\hbar\omega$ and neglect the quantum pressure terms as in [14] in the time evolution of the velocity fields:

$$\partial_t\rho_\varepsilon+\mathrm{div}(\rho_\varepsilon\mathbf{v}_\varepsilon)=0 \quad (71)$$

$$\partial_t\mathbf{v}_\varepsilon+\frac{1}{2}\,\mathbf{grad}\;v_\varepsilon^2=-\frac{1}{m}\,\mathbf{grad}\;[U(\mathbf{r})+\rho_\varepsilon g_{\varepsilon\varepsilon}+\rho_{\varepsilon'}g_{\varepsilon\varepsilon'}]. \quad (72)$$

At this point we introduce the deviations of the densities and velocity fields from their initial values:

$$\rho_\varepsilon(t)=\rho_\varepsilon(0)+\delta\rho_\varepsilon(t) \quad (73)$$

$$\mathbf{v}_\varepsilon(t)=\mathbf{v}_\varepsilon(0)+\delta\mathbf{v}_\varepsilon(t) \quad (74)$$

where the initial values are given by:

$$\rho_\varepsilon(t=0)=N_\varepsilon|\phi_0|^2(N) \quad (75)$$

$$\mathbf{v}_\varepsilon(t=0)=\mathbf{0}. \quad (76)$$

By expanding equations (71, 72) to first order in the small quantities $\delta\rho_\varepsilon,\delta\mathbf{v}_\varepsilon$, we obtain:

$$\partial_t\delta\rho_\varepsilon+\mathrm{div}[N_\varepsilon|\phi_0|^2\delta\mathbf{v}_\varepsilon]=0 \quad (77)$$

$$\partial_t\delta\mathbf{v}_\varepsilon+\frac{1}{m}\,\mathbf{grad}\;[\delta\rho_\varepsilon g_{\varepsilon\varepsilon}+\delta\rho_{\varepsilon'}g_{\varepsilon\varepsilon'}]=-\frac{1}{m}\,\mathbf{grad}\;[|\phi_0|^2](N_\varepsilon g_{\varepsilon\varepsilon}+N_{\varepsilon'}g_{\varepsilon\varepsilon'}-Ng_\mathrm{aa}). \quad (78)$$

By taking the first time derivative of equation (77) we eliminate the velocity field and we get:

$$\partial_t^2\delta\rho_\varepsilon+\sum_{\varepsilon'}M_{\varepsilon\varepsilon'}S[\delta\rho_{\varepsilon'}]+\sigma_\varepsilon=0. \quad (79)$$

The source terms of these inhomogeneous equations are:

$$\sigma_\varepsilon=-\frac{N_\varepsilon}{m}\mathrm{div}[|\phi_0|^2\;\mathbf{grad}\;|\phi_0|^2](N_\varepsilon g_{\varepsilon\varepsilon}+N_{\varepsilon'}g_{\varepsilon\varepsilon'}-Ng_\mathrm{aa}). \quad (80)$$

The homogeneous part of equation (79) involves the 2×2 matrix M:

$$M=\frac{1}{Ng_\mathrm{aa}}\begin{pmatrix}N_\mathrm{a}g_\mathrm{aa} & N_\mathrm{a}g_\mathrm{ab}\\ N_\mathrm{b}g_\mathrm{ab} & N_\mathrm{b}g_\mathrm{bb}\end{pmatrix} \quad (81)$$

and the Stringari operator defined in equation (53). In order to solve equation (79) we introduce the eigenvectors $\mathbf{e}_\pm$ of the matrix M with corresponding eigenvalues $g_\pm$. Consistently with our previous approximations, we calculate, to leading order in the differences between the coupling constants, the eigenvalues:

$$g_+\simeq g_\mathrm{aa} \quad (82)$$

$$g_-\simeq\frac{N_\mathrm{a}N_\mathrm{b}}{N^2}(g_\mathrm{aa}+g_\mathrm{bb}-2g_\mathrm{ab}) \quad (83)$$

and the components of $(\delta\rho_\mathrm{a}, \delta\rho_\mathrm{b})$ on the eigenvectors of M:

$$\delta\rho_+ \simeq \delta\rho_\mathrm{a} + \delta\rho_\mathrm{b} \quad (84)$$

$$\delta\rho_- \simeq \frac{N_\mathrm{b}}{N}\delta\rho_\mathrm{a} - \frac{N_\mathrm{a}}{N}\delta\rho_\mathrm{b}. \quad (85)$$

For those linear combinations we get the decoupled equations:

$$\partial_t^2 \delta\rho_\pm + \frac{g_\pm}{g_\mathrm{aa}} S[\delta\rho_\pm] + \sigma_\pm = 0. \quad (86)$$

To study the dynamics of the system we expand $\rho_\pm$ and the source terms $\sigma_\pm$ on the eigenmodes of the Stringari operator. It turns out that the source terms are simply proportional to the breathing mode $\alpha_{q=5}$ already introduced in equation (57). The solution of equation (86) with the initial conditions $\delta\rho_\pm = \partial_t \delta\rho_\pm = 0$ is then:

$$\delta\rho_\pm(\mathbf{r}, t) = N|\phi_0(\mathbf{0})|^2 \mathcal{A}_\pm \frac{g_\mathrm{aa}}{g_\pm}[1 - \cos \Omega_\pm t]\alpha_{q=5}(r) \quad (87)$$

with eigenfrequencies and amplitudes given by:

$$\Omega_\pm = \left(\frac{5g_\pm}{g_\mathrm{aa}}\right)^{1/2} \omega \quad (88)$$

$$\mathcal{A}_+ = \frac{N_\mathrm{a}^2 g_\mathrm{aa} + N_\mathrm{b}^2 g_\mathrm{bb} + 2N_\mathrm{a}N_\mathrm{b}g_\mathrm{ab}}{N^2 g_\mathrm{aa}} - 1 \quad (89)$$

$$\mathcal{A}_- = \frac{N_\mathrm{a}N_\mathrm{b}}{N^2}\left[\frac{N_\mathrm{a}g_\mathrm{aa} + N_\mathrm{b}g_\mathrm{ab} - N_\mathrm{b}g_\mathrm{bb} - N_\mathrm{a}g_\mathrm{ab}}{Ng_\mathrm{aa}}\right]. \quad (90)$$

We note that when the numbers of atoms $N_\mathrm{a,b}$ satisfy the breathe-together condition (Eq. (26)) the amplitude $\mathcal{A}_-$ vanishes as expected, since $\delta\rho_- \equiv 0$ in this case.

4.2 Validity of the linear approximation

In order for our linearized treatment to be valid the deviations $\delta\rho_\pm$ should remain small as compared to the initial densities. A first necessary condition to be satisfied is that the eigenfrequencies $\Omega_\pm$ should be real. This imposes the positivity of the matrix M, ensured by the positivity of its determinant:

$$g_\mathrm{ab}^2 \le g_\mathrm{aa}g_\mathrm{bb}. \quad (91)$$

This condition is known in the case of homogeneous mixtures of condensates as a stability condition against demixing [23]. To the leading order in the difference between the coupling constants, the condition (91) is equivalent to $g_\mathrm{aa} + g_\mathrm{bb} - 2g_\mathrm{ab} > 0$.

We note at this point that the amplitude $\mathcal{A}_-/g_-$ in the expression for $\delta\rho_-$ is a ratio of two small numbers. When this ratio is large the system can evolve far from its initial state even in the stable case $g_- > 0$: numerical solutions of the Gross-Pitaevskii equations confirm this expectation, showing the formation of a crater at the center of one of the condensates. We therefore have to impose a second condition:

$$\left|\mathcal{A}_\pm \frac{g_\mathrm{aa}}{g_\pm}\right| \ll 1. \quad (92)$$

Finally the present treatment is based on the classical hydrodynamic approximation; by including the quantum pressure terms in the hydrodynamic equation for the velocity field one can show that this imposes on the eigenfrequencies Ω_-:

$$\frac{\hbar\omega^2}{\mu} \ll \Omega_- \quad (93)$$

(see also Appendix C). This condition can be violated even in the Thomas-Fermi limit, when the g_- eigenvalue almost vanishes. In this case one has to include the quantum pressure terms; the decoupling property of $\delta\rho_\pm$ is unaffected; for the evolution of $\delta\rho_-$ similar results as in equation (87) are obtained; we find *e.g.* $\Omega_- \simeq 63\hbar\omega^2/8\mu$.

4.3 Phase dynamics

We assume that all the conditions for the validity of the linearized treatment are satisfied so that we can proceed to the analysis of the relative phase dynamics. To this aim we write the equation of evolution for the phases θ_ε of the condensate wavefunctions ϕ_ε in the classical hydrodynamic approximation:

$$\partial_t \theta_\varepsilon + \frac{\hbar}{2m}(\mathbf{grad}\ \theta_\varepsilon)^2 = -[U + g_{\varepsilon\varepsilon}\rho_\varepsilon + g_{\varepsilon\varepsilon'}\rho_{\varepsilon'}]/\hbar. \quad (94)$$

The equations for the velocity fields previously given are simply the gradient of equation (94). By linearizing equation (94) around the initial state $\theta_\varepsilon = 0$ we obtain for the relative phase:

$$\hbar\partial_t(\theta_\mathrm{a} - \theta_\mathrm{b}) \simeq -|\phi_0|^2(N_\mathrm{a}g_\mathrm{aa} + N_\mathrm{b}g_\mathrm{ab} - N_\mathrm{b}g_\mathrm{bb} - N_\mathrm{a}g_\mathrm{ab}) + (g_\mathrm{ab} - g_\mathrm{aa})\delta\rho_\mathrm{a} + (g_\mathrm{bb} - g_\mathrm{ab})\delta\rho_\mathrm{b}. \quad (95)$$

The right hand side of this equation is a sum of terms constant in time and of oscillatory functions of time. The function $\theta_\mathrm{a} - \theta_\mathrm{b}$ then has two components: an oscillating component and a component diverging linearly with time which will dominate for long times. By using the result equation (87) and the Thomas-Fermi approximation for $|\phi(0)|^2$ (Eq. (43)) we can calculate the time diverging component and we obtain to leading order in the g's difference:

$$\theta_\mathrm{a} - \theta_\mathrm{b} \sim -\frac{2\mu}{5Ng_\mathrm{aa}}[N_\mathrm{a}g_\mathrm{aa} - N_\mathrm{b}g_\mathrm{bb} + (N_\mathrm{b} - N_\mathrm{a})g_\mathrm{ab}]\ t/\hbar. \quad (96)$$

We now use equations (17, 20) to obtain:

$$\chi_\mathrm{d} \sim -\frac{1}{2}\left(\frac{\mathrm{d}\mu}{\mathrm{d}N}\right)_{N=\bar{N}} \frac{g_\mathrm{aa} + g_\mathrm{bb} - 2g_\mathrm{ab}}{g_\mathrm{aa}}\ t/\hbar \quad (97)$$

$$\chi_\mathrm{s} + (|c_\mathrm{a}|^2 - |c_\mathrm{b}|^2)\chi_\mathrm{d} \sim -\frac{2}{5g_\mathrm{aa}}\left(\frac{\mathrm{d}\mu}{\mathrm{d}N}\right)_{N=\bar{N}} \times (|c_\mathrm{a}|^2 g_\mathrm{aa} + |c_\mathrm{b}|^2 g_\mathrm{ab} - |c_\mathrm{b}|^2 g_\mathrm{bb} - |c_\mathrm{a}|^2 g_\mathrm{ab})\ t/\hbar \quad (98)$$

where we introduced the derivative of the chemical potential with respect to the total number of particle $(\mathrm{d}\mu/\mathrm{d}N)(N = \bar{N}) \simeq (2/5)\bar{\mu}/\bar{N}$ in the Thomas-Fermi limit. As we already found in the particular case of the breathe-together solution the constants χ_{d} and χ_{s} governing the relative phase collapse are highly reduced for close g's with respect to the case of non mutually interacting condensates.

4.4 Physical interpretation of the results

We now show that all the previous results of this section can be interpreted in terms of small oscillations of the condensates around the steady state.

Let us introduce the steady state densities $\rho_\varepsilon^{\mathrm{st}}$ for the condensates with N_{a} particles in a and N_{b} particles in b. As we are in the case of quasi complete spatial overlap between the two condensates we can use the Thomas-Fermi approximation to determine these densities:

$$\mu_{\mathrm{a}} - U = \rho_{\mathrm{a}}^{\mathrm{st}} g_{\mathrm{aa}} + \rho_{\mathrm{b}}^{\mathrm{st}} g_{\mathrm{ab}} \tag{99}$$

$$\mu_{\mathrm{b}} - U = \rho_{\mathrm{a}}^{\mathrm{st}} g_{\mathrm{ab}} + \rho_{\mathrm{b}}^{\mathrm{st}} g_{\mathrm{bb}} \tag{100}$$

where μ_ε are the chemical potentials in steady state. We rewrite these equations in terms of the deviations $\delta\rho_\varepsilon^{\mathrm{st}}$ of the steady-state densities from the initial state densities $N_\varepsilon|\phi_0|^2$ and in terms of the deviations $\delta\mu_\varepsilon$ of the chemical potentials from μ defined in equation (1):

$$\delta\mu_{\mathrm{a}} = (N_{\mathrm{a}}g_{\mathrm{aa}} + N_{\mathrm{b}}g_{\mathrm{ab}} - Ng_{\mathrm{aa}})|\phi_0|^2 + \delta\rho_{\mathrm{a}}^{\mathrm{st}} g_{\mathrm{aa}} + \delta\rho_{\mathrm{b}}^{\mathrm{st}} g_{\mathrm{ab}} \tag{101}$$

$$\delta\mu_{\mathrm{b}} = (N_{\mathrm{b}}g_{\mathrm{bb}} + N_{\mathrm{a}}g_{\mathrm{ab}} - Ng_{\mathrm{aa}})|\phi_0|^2 + \delta\rho_{\mathrm{a}}^{\mathrm{st}} g_{\mathrm{ab}} + \delta\rho_{\mathrm{b}}^{\mathrm{st}} g_{\mathrm{bb}}. \tag{102}$$

Using the fact that the spatial integral of $\delta\rho_\varepsilon$ vanishes, we get from integration of equations (101, 102) over the volume of $|\phi_0|^2$ the approximate relations:

$$\delta\mu_{\mathrm{a}} = \frac{2\mu}{5Ng_{\mathrm{aa}}}(N_{\mathrm{a}}g_{\mathrm{aa}} + N_{\mathrm{b}}g_{\mathrm{ab}} - Ng_{\mathrm{aa}}) \tag{103}$$

$$\delta\mu_{\mathrm{b}} = \frac{2\mu}{5Ng_{\mathrm{aa}}}(N_{\mathrm{b}}g_{\mathrm{bb}} + N_{\mathrm{a}}g_{\mathrm{ab}} - Ng_{\mathrm{aa}}). \tag{104}$$

We can therefore check that the relative phase of the condensates in steady state, given by $\theta_{\mathrm{a}}^{\mathrm{st}} - \theta_{\mathrm{b}}^{\mathrm{st}} = -\mathrm{i}(\delta\mu_{\mathrm{a}} - \delta\mu_{\mathrm{b}})t/\hbar$, evolves as in equation (96). The phase decoherence properties of the evolving mixture are then essentially the same as in steady state.

Moreover we now show that the average $\langle\delta\rho_\varepsilon\rangle$ of $\delta\rho_\varepsilon$ over the oscillations at frequencies $\Omega_\pm$ coincide with $\delta\rho_\varepsilon^{\mathrm{st}}$. First, by averaging equation (77) over time we find that the velocity fields have a vanishing time average [24]. Second, we average equation (78) over time; we find equations for the spatial gradient of $\langle\delta\rho_\varepsilon\rangle$, which coincide with the spatial gradient of equations (101, 102), so that $\langle\delta\rho_\varepsilon\rangle = \delta\rho_\varepsilon^{\mathrm{st}}$ [25].

5 Discussion of the JILA case

In the JILA experiment the values of the three coupling constants between the atoms are known with good precision; they are in the ratio [10]:

$$g_{\mathrm{aa}} : g_{\mathrm{ab}} : g_{\mathrm{bb}} = 1.03 : 1 : 0.97. \tag{105}$$

No breathe-together solution exists in this case, as g_{ab} lies within g_{aa} and g_{bb}. Experimentally half of the particles are in the state a so that $|c_{\mathrm{a}}|^2 = |c_{\mathrm{b}}|^2 = 1/2$, and the mean total number of particles is $\bar{N} = 5 \times 10^5$. Although the coupling constants are close, the linearized treatment presented in Section 4 does not apply either, because condition (92) is violated. It is actually found experimentally that the two condensates evolve far from the initial state, with formation of a crater in the a condensate while the b condensate becomes more confined at the center of the trap; eventually the condensates separate in some random direction [10].

To avoid the crater formation and trigger the spatial separation of the two condensates in a reproducible direction a small spatial shift is applied to the trapping potential of one of the two states. The two condensates separate, with a relative motion exhibiting strongly damped oscillations [10]. The system then reaches a steady state that still exhibits phase coherence, up to times on the order of 150 ms after the phase state preparation [9].

5.1 Time dependent calculations

We have already studied in [15] the damping of the relative motion between the condensates, by numerical integration of the coupled Gross-Pitaevskii equations (Eq. (9)). The agreement with the experimental results of [10] is qualitatively good, although the damping in the theory is weaker and incomplete, small oscillations of the condensate wavefunctions remaining undamped even at long times.

We have applied the formalism of Section 2 by numerically integrating the Gross-Pitaevskii equations for the parameters of the JILA experiment. The coefficients $\chi_{\mathrm{s}}, \chi_{\mathrm{d}}$, now complicated functions of time and space, are obtained by evolving wavefunctions with slightly different numbers of atoms in a and b. In order to facilitate the comparison with the experiments, in which the x-integrated atomic density $\bar{\rho}_{\mathrm{a}}(y, z)$ in the internal state a is measured after the $\pi/2$ pulse applied at time τ, we calculated the following contrast:

$$\begin{aligned} C_{\mathrm{JILA}}(y,z) &= \frac{\max_\delta \bar{\rho}_{\mathrm{a}} - \min_\delta \bar{\rho}_{\mathrm{a}}}{\max_\delta \bar{\rho}_{\mathrm{a}} + \min_\delta \bar{\rho}_{\mathrm{a}}} \\ &= \frac{2|\int \mathrm{d}x\ \langle\hat{\psi}_{\mathrm{a}}^\dagger(\tau^-)\hat{\psi}_{\mathrm{b}}(\tau^-)\rangle^{\mathrm{Gauss}}|}{\sum_{\varepsilon=\mathrm{a,b}} \int \mathrm{d}x\ \bar{N}_\varepsilon|\bar{\phi}_\varepsilon|^2(\tau^-)} \end{aligned} \tag{106}$$

where the interference term (15) is averaged over a Gaussian distribution of the total number of particles with a standard deviation ΔN. A direct comparison with the experiment would require the inclusion of the 22 ms ballistic

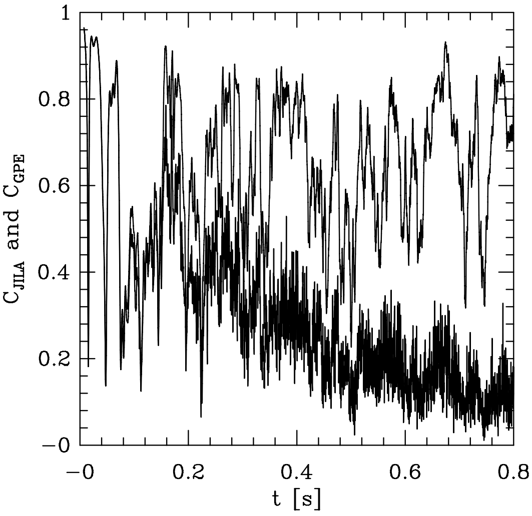

Fig. 2. For the parameters of the JILA experiment (not including the 22 ms ballistic expansion), phase contrasts $C_{\rm JILA}$ (lower curve) and $C_{\rm GPE}$ (upper curve) defined in the text, at $y = z = 0$, as function of time in seconds, for the evolving binary mixture, with $\Delta N = 0.08\bar{N}$.

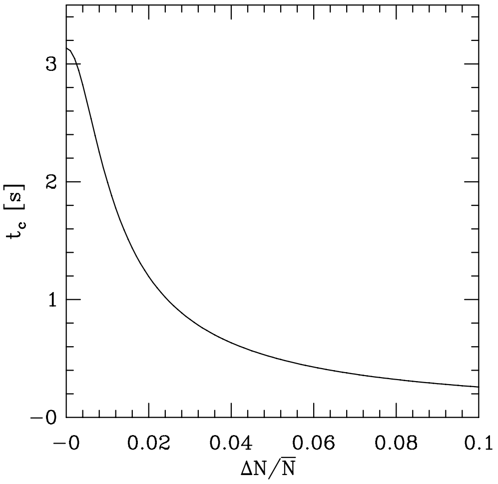

Fig. 3. For the parameters of the JILA experiment (except the 22 ms ballistic expansion), collapse time $t_{\rm c}$ for $C_{\rm JILA}$ at $y = z = 0$ as a function of $\Delta N/\bar{N}$ for zero temperature steady state condensates in the shifted traps.

expansion, not included in the present numerical calculations.

Our numerical result for $C_{\rm JILA}$ at the center of the trap for the species a, $y = z = 0$, is presented in Figure 2, for Gaussian fluctuations in the total number of particles $\Delta N/\bar{N} = 8\%$ corresponding to the JILA experiment [26], together with the pure Gross-Pitaevskii prediction $C_{\rm GPE}$ obtained by setting all the χ's to 0. The Gross-Pitaevskii prediction oscillates around $\langle C_{\rm GPE}\rangle = 0.63$. On the contrary the result of the more complete calculation including fluctuations in the relative and total number of particles exhibits a damping of the contrast, that we have fitted by convenience with the formula $C_{\rm JILA} = C_0 {\rm e}^{-\gamma t}$; we obtain $C_0 \simeq \langle C_{\rm GPE}\rangle$ and $\gamma^{-1} = 0.42$ s.

Note the oscillatory aspect of the curves in Figure 2. More understanding of the structure of the condensate wavefunctions given by equation (9) is required as this point: as detailed in [15] $\bar{\phi}_\varepsilon$ is a sum of a smooth part, performing oscillations with frequencies expected to be close to eigenfrequencies of the steady state condensates [27], and of a noisy quasi-stochastic part. The slow oscillatory structure evident on $C_{\rm GPE}$ comes from this smooth oscillating part of the wavefunctions.

We have also considered the ideal case of a well-defined total number of particles. The numerical prediction for the contrast $C_{\rm JILA}$ in this case corresponds to a very long lived phase coherence: after a time of 1 s, the contrast is still very close to the pure Gross-Pitaevskii prediction.

5.2 Steady state calculations and effect of particle losses

As the wavefunctions at long times perform mainly oscillations around the steady state we have also tried a much simpler steady state calculation (see Sect. 2.5). During the collapse time the contrast $C_{\rm JILA}$ is a Gaussian in time (Eq. (22)), with an initial value 0.958 and with a half-width $t_{\rm c}$ at the relative height ${\rm e}^{-1/2}$. We plot in Figure 3 the variation of $t_{\rm c}$ as function of the standard deviation ΔN. As we find $\chi_{\rm s}/t = -7.7 \times 10^{-5}$ s^{-1} and $\chi_{\rm d}/t = -4.5 \times 10^{-4}$ s^{-1}, one has $|\chi_{\rm s}| \simeq |\chi_{\rm d}|/6$, so that relatively high values of ΔN are required to observe a significant effect of the fluctuations of the total number of particles on phase decoherence. For $\Delta N = 0.08\bar{N}$ the phase decoherence time is $t_{\rm c} = 0.32$ s, close to the result of the time-dependent calculation of Figure 2. Note that for such a high value of $\Delta N/\bar{N}$ the decay of the phase contrast in equation (22) is essentially due to the first exponential factor accounting for the smearing of the phase by fluctuations of the total number of particles, the spreading of the phase for a fixed number of particles being very small ($\bar{N}\chi_{\rm d}^2(t_{\rm c})/2 \simeq 0.005$).

We now briefly consider the issue of losses of particles. An intrinsic source of losses in the JILA experiment are the inelastic collisions between a atoms and b atoms, resulting in the simultaneous loss of two particles. We estimate the mean number $\langle\delta N\rangle$ of lost particles from the rate constant K_2 for binary inelastic collisions between the states $|F = 1, m = -1\rangle$ and $|F = 2, m = 2\rangle$ [28] and from a numerical calculation of the overlap integral $\int {\rm d}^3\mathbf{r}\, |\bar{\phi}_{\rm a}|^2 |\bar{\phi}_{\rm b}|^2$. For the JILA parameters we find $\langle\delta N\rangle/\bar{N} = 0.04$ at time $t_{\rm c} = 0.32$ s. One could then naively expect the effect of losses on phase coherence to be comparable with the effect of fluctuations of N.

To test this naive expectation we use the following simple model, inspired by the two-mode model developed in [8], and focusing on the effect of the losses on the drift velocity $v(N)$ of the relative phase of the two condensates given in equation (21). Imagine that the system has initially $\bar{N}$ condensate atoms and that k binary inelastic collisions have taken place at times $t_1 < \ldots < t_k$ between time 0 and time t. The shift of the relative phase during t is then given by:

$$\Theta = \int_0^t \mathrm{d}\tau \; v(N(\tau)) = v(\bar{N})t_1 + v(\bar{N}-2)(t_2-t_1) + \ldots + v(\bar{N}-2k)(t-t_k). \quad (107)$$

As we do in [8] we assume a constant mean number of collisions λ per unit of time and we average the phase factor $\mathrm{e}^{\mathrm{i}\Theta}$ multiplying the interference term $\langle \hat{\psi}_\mathrm{b}^\dagger \hat{\psi}_\mathrm{a} \rangle$ over the probability distribution of the times $t_1, \ldots, t_k$ and of the number of loss events k,

$$P_t(t_1, \ldots, t_k; k) = \lambda^k \mathrm{e}^{-\lambda t} \quad (108)$$

to obtain:

$$|\langle \mathrm{e}^{\mathrm{i}\Theta} \rangle| = \exp\{-\langle k \rangle [1 - \sin(2\chi_\mathrm{s})/(2\chi_\mathrm{s})]\} \quad (109)$$

$$\simeq \exp\left[-\frac{2}{3}\langle k \rangle \chi_\mathrm{s}^2\right] \quad \text{for } |\chi_\mathrm{s}| \ll 1 \quad (110)$$

where $2\langle k \rangle = 2\lambda t = \langle \delta N \rangle$ is the mean number of lost particles during t. At time $t = t_\mathrm{c} = 0.32$ s the corresponding modulus of the averaged phase factor is on the order of $[1-4\times10^{-6}]$, very close to one: particle losses have a negligible effect on the phase coherence at the considered time t_c, even if $\langle \delta N \rangle$ and ΔN have the same order of magnitude.

Actually an inspection of the χ_s dependent factor in equation (22) and of equation (110) reveals that these equations have the same structure; replacing in equation (22) the variance ΔN^2 of the total number of particles by the variance Δk^2 of the number of loss events ($\Delta k^2 = \langle k \rangle$ as k obeys a Poisson law) one recovers equation (110) up to a numerical factor inside the exponential. For equally large values of ΔN and $\langle k \rangle$ the effect of losses on phase coherence is less important than that of fluctuations of N because $\Delta k^2 = \langle k \rangle \ll \Delta N^2$.

We have also investigated another source of losses, the collisions of condensate atoms with the background gas of the cell. Assuming a lifetime of the particles in the cell of 250 s as in [29] we find as well that this loss mechanism has a negligible effect on the phase coherence for a time $t_\mathrm{c} = 0.32$ s.

6 Conclusion and perspectives

We have extended previous treatments of the phase dynamics of Bose-Einstein condensates at zero temperature to the case of mutually interacting and dynamically evolving binary mixtures of condensates, for a measurement scheme of the phase coherence inspired by the JILA experiment.

We have first applied this extended formalism to the interesting breathe-together solution of the Gross-Pitaevskii equations, in which the two condensates oscillate in phase, remaining always exactly spatially superimposed. The analytical results for the phase show that a dramatic increase of the phase coherence time can be obtained for close coupling constants $g_\mathrm{aa}, g_\mathrm{ab}, g_\mathrm{bb}$ describing the elastic interactions between a atoms and b atoms.

We have also treated analytically the case of close g's, in the absence of demixing instability. Basically the phase collapse is identical to the steady state case for the two mutually interacting condensates.

Finally, we have investigated numerically the more difficult case of JILA. We find a collapse time of the phase on the order of 0.4 s, both by a dynamical and a steady state calculation, in the case of Gaussian fluctuations of the total number of particles, corresponding to $\Delta N/\bar{N} = 8\%$. This result for the collapse time is significantly larger than the experimental results (no phase coherence measured after 150 ms). We have also estimated in a simple way the effect of collisional losses on phase coherence in the JILA experiment.

A possible extension of this work could include the effect of the presence of a thermal component in the experiment.

Part of this work (the breathe-together solution) would have not been possible without the contribution of G. Shlyapnikov, J. Dalibard and P. Fedichev. We thank A. Leggett, Y. Kagan for very useful discussions on the role of fluctuations in the total number of particles. We thank Ralph Dum for help in the numerical calculations. A.S. acknowledges financial support from the European Community (TMR individual research grant).

Appendix A: Phase correction to the Gross-Pitaevskii prediction

We consider the evolution of the Fock state $|N_\mathrm{a}:\phi_\mathrm{a}(0), N_\mathrm{b}:\phi_\mathrm{b}(0)\rangle$ (with $N_\mathrm{a,b}$ particles in the internal state a, b). The model Hamiltonian we consider contains the one-body Hamiltonians $\mathcal{H}_\varepsilon$ and elastic interactions terms:

$$H = \int \mathrm{d}^3\mathbf{r} \sum_{\varepsilon=\mathrm{a,b}} \hat{\psi}_\varepsilon^\dagger \mathcal{H}_\varepsilon \hat{\psi}_\varepsilon + \frac{1}{2} g_\mathrm{aa} \hat{\psi}_\mathrm{a}^\dagger \hat{\psi}_\mathrm{a}^\dagger \hat{\psi}_\mathrm{a} \hat{\psi}_\mathrm{a} + \frac{1}{2} g_\mathrm{bb} \hat{\psi}_\mathrm{b}^\dagger \hat{\psi}_\mathrm{b}^\dagger \hat{\psi}_\mathrm{b} \hat{\psi}_\mathrm{b} + g_\mathrm{ab} \hat{\psi}_\mathrm{b}^\dagger \hat{\psi}_\mathrm{a}^\dagger \hat{\psi}_\mathrm{a} \hat{\psi}_\mathrm{b} \quad (\mathrm{A}.1)$$

where $\hat{\psi}_\varepsilon$ is the atomic field operator in the internal state ε.

We use the Hartree-Fock type ansatz for the N-body state vector:

$$|\Psi\rangle = \mathrm{e}^{-\mathrm{i}A(t)/\hbar} |N_\mathrm{a}:\phi_\mathrm{a}(t), N_\mathrm{b}:\phi_\mathrm{b}(t)\rangle. \quad (\mathrm{A}.2)$$

A variational formulation of the Hamiltonian equation

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\Psi\rangle = H |\Psi\rangle \quad (\mathrm{A}.3)$$

leads to the Gross-Pitaevskii equations for $\phi_\varepsilon(t)$, given in equation (9), up to the undetermined phase factor A corresponding formally to a time dependent Lagrange multiplier ensuring the conservation of the norm of $|\Psi\rangle$. To determine this phase factor A, we multiply equation (A.3) on the left by the bra $\langle\Psi|$; we obtain:

$$\dot{A} + \mathrm{i}\hbar\langle N_\mathrm{a} : \phi_\mathrm{a}(t), N_\mathrm{b} : \phi_\mathrm{b}(t)|\frac{\mathrm{d}}{\mathrm{d}t}|N_\mathrm{a} : \phi_\mathrm{a}(t), N_\mathrm{b} : \phi_\mathrm{b}(t)\rangle = \langle\Psi|H|\Psi\rangle. \quad (\mathrm{A}.4)$$

The scalar products are calculated in second quantized formalism, *e.g.* we find:

$$\langle N_\mathrm{a} : \phi_\mathrm{a}(t), N_\mathrm{b} : \phi_\mathrm{b}(t)|\frac{\mathrm{d}}{\mathrm{d}t}|N_\mathrm{a} : \phi_\mathrm{a}(t), N_\mathrm{b} : \phi_\mathrm{b}(t)\rangle = \sum_\varepsilon N_\varepsilon\langle\phi_\varepsilon|\frac{\mathrm{d}}{\mathrm{d}t}|\phi_\varepsilon\rangle. \quad (\mathrm{A}.5)$$

We finally arrive at equation (11).

Appendix B: Derivation of the interference term

When the N-body state vector is initial a phase state (2) and if one assumes that the Fock states evolve according to equation (8) one gets the following expression for the interference term between the two condensates:

$$\begin{aligned}\langle\hat{\psi}_\mathrm{b}^\dagger\hat{\psi}_\mathrm{a}\rangle_N = {} & c_\mathrm{a}c_\mathrm{b}^*\sum_{N_\mathrm{a}=1}^{N}\frac{N!}{(N_\mathrm{a}-1)!N_\mathrm{b}!}|c_\mathrm{a}|^{2(N_\mathrm{a}-1)}|c_\mathrm{b}|^{2N_\mathrm{b}} \\ & \times \phi_\mathrm{a}(N_\mathrm{a},N_\mathrm{b})\phi_\mathrm{b}^*(N_\mathrm{a}-1,N_\mathrm{b}+1) \\ & \times \mathrm{e}^{\mathrm{i}[A(N_\mathrm{a}-1,N_\mathrm{b}+1)-A(N_\mathrm{a},N_\mathrm{b})]/\hbar} \\ & \times [\langle\phi_\mathrm{a}(N_\mathrm{a}-1,N_\mathrm{b}+1)|\phi_\mathrm{a}(N_\mathrm{a},N_\mathrm{b})\rangle]^{N_\mathrm{a}-1} \\ & \times [\langle\phi_\mathrm{b}(N_\mathrm{a}-1,N_\mathrm{b}+1)|\phi_\mathrm{b}(N_\mathrm{a},N_\mathrm{b})\rangle]^{N_\mathrm{b}}\end{aligned} \quad (\mathrm{B}.1)$$

where $N_\mathrm{b} = N - N_\mathrm{a}$. In the large N limit, we expand to first order the effect of shifts of N_ε by unity in the last three lines of the previous equation:

$$\phi_\mathrm{a}(N_\mathrm{a}-1,N_\mathrm{b}+1) \simeq \phi_\mathrm{a}(N_\mathrm{a}-1,N_\mathrm{b})+\partial_{N_\mathrm{b}}\phi_\mathrm{a}(N_\mathrm{a}-1,N_\mathrm{b}) \quad (\mathrm{B}.2)$$

$$\phi_\mathrm{a}(N_\mathrm{a},N_\mathrm{b}) \simeq \phi_\mathrm{a}(N_\mathrm{a}-1,N_\mathrm{b})+\partial_{N_\mathrm{a}}\phi_\mathrm{a}(N_\mathrm{a}-1,N_\mathrm{b}) \quad (\mathrm{B}.3)$$

$$A(N_\mathrm{a}-1,N_\mathrm{b}+1) \simeq A(N_\mathrm{a}-1,N_\mathrm{b})+\partial_{N_\mathrm{b}}A(N_\mathrm{a}-1,N_\mathrm{b}) \quad (\mathrm{B}.4)$$

$$A(N_\mathrm{a},N_\mathrm{b}) \simeq A(N_\mathrm{a}-1,N_\mathrm{b})+\partial_{N_\mathrm{a}}A(N_\mathrm{a}-1,N_\mathrm{b}). \quad (\mathrm{B}.5)$$

We then get:

$$\begin{aligned}\langle\hat{\psi}_\mathrm{b}^\dagger\hat{\psi}_\mathrm{a}\rangle_N = {} & c_\mathrm{a}c_\mathrm{b}^*\sum_{N_\mathrm{a}=1}^{N}\frac{N!}{(N_\mathrm{a}-1)!N_\mathrm{b}!}|c_\mathrm{a}|^{2(N_\mathrm{a}-1)}|c_\mathrm{b}|^{2N_\mathrm{b}} \\ & \times \phi_\mathrm{a}(N_\mathrm{a},N_\mathrm{b})\phi_\mathrm{b}^*(N_\mathrm{a}-1,N_\mathrm{b}+1) \\ & \times \mathrm{e}^{\mathrm{i}\Theta(N_\mathrm{a}-1,N_\mathrm{b})}\end{aligned} \quad (\mathrm{B}.6)$$

where we have introduced the real quantity:

$$\begin{aligned}\Theta(N_\mathrm{a},N_\mathrm{b}) = {} & \frac{1}{\hbar}(\partial_{N_\mathrm{b}}-\partial_{N_\mathrm{a}})A(N_\mathrm{a},N_\mathrm{b}) \\ & + \mathrm{i}\sum_\varepsilon N_\varepsilon\langle\phi_\varepsilon(N_\mathrm{a},N_\mathrm{b})|(\partial_{N_\mathrm{b}}-\partial_{N_\mathrm{a}})|\phi_\varepsilon(N_\mathrm{a},N_\mathrm{b})\rangle.\end{aligned} \quad (\mathrm{B}.7)$$

We calculate the time derivative of $\Theta(N_\mathrm{a},N_\mathrm{b})$ using the Gross-Pitaevskii equations (Eq. (9)). After lengthy calculations we find

$$\dot{\Theta}(N_\mathrm{a},N_\mathrm{b}) = 0. \quad (\mathrm{B}.8)$$

In the Gedanken experiment considered in this paper, the initial wavefunctions $\phi_\varepsilon(t=0)$ depend only on $N_\mathrm{a}+N_\mathrm{b}$ so that they have a vanishing derivative $\partial_{N_\mathrm{b}}-\partial_{N_\mathrm{a}}$, and we take initially $A=0$; this leads to $\Theta\equiv 0$. The same conclusion holds if the initial wavefunctions are real.

Appendix C: Approximate evolution in the Thomas-Fermi limit

After the gauge and scale transforms equations (40, 41), the equations of evolution for $\tilde{\bar{\phi}}$ and $\delta\tilde{\varphi}_\mathrm{d}$ read:

$$\mathrm{i}\hbar\partial_t\tilde{\bar{\phi}} = -\frac{\hbar^2}{2m\lambda^2}\Delta\tilde{\bar{\phi}} + \frac{g}{g_\mathrm{aa}\lambda^3}\left[U(\mathbf{r})+\bar{N}g_\mathrm{aa}|\tilde{\bar{\phi}}|^2-\bar{\mu}\right]\tilde{\bar{\phi}} \quad (\mathrm{C}.1)$$

$$\begin{aligned}\mathrm{i}\hbar\partial_t\delta\tilde{\varphi}_\mathrm{d} = {} & -\frac{\hbar^2}{2m\lambda^2}\Delta\delta\tilde{\varphi}_\mathrm{d} \\ & + \frac{g}{g_\mathrm{aa}\lambda^3}\left[U(\mathbf{r})+\bar{N}g_\mathrm{aa}|\tilde{\bar{\phi}}|^2-\bar{\mu}\right]\delta\tilde{\varphi}_\mathrm{d} \\ & + \frac{1}{\lambda^3}N_\mathrm{b}(g_\mathrm{bb}-g_\mathrm{ab})(|\tilde{\bar{\phi}}|^2\delta\tilde{\varphi}_\mathrm{d}+\tilde{\bar{\phi}}^2\delta\tilde{\varphi}_\mathrm{d}^*).\end{aligned} \quad (\mathrm{C}.2)$$

In the Thomas-Fermi limit the terms involving the Laplacian are small; if we neglect them we get for the time derivatives of the α and β variables defined in equations (44, 45):

$$\mathrm{i}\hbar\partial_t\alpha = 0 \quad (\mathrm{C}.3)$$

$$\mathrm{i}\hbar\partial_t\beta = \frac{1}{\lambda^3}N_\mathrm{a}(g_\mathrm{aa}-g_\mathrm{ab})\alpha. \quad (\mathrm{C}.4)$$

The variable α has actually been defined in a way to obtain zero on the right hand side of equation (C.3).

The first equation (C.3) is not an acceptable approximation for the evolution of α, we therefore include in $\partial_t\alpha$ the contribution of the Laplacian terms:

$$\mathrm{i}\hbar\partial_t\alpha = -\frac{\hbar^2}{2m\lambda^2}\,\mathrm{div}\left\{\alpha\left[\frac{\mathbf{grad}\,\tilde{\bar{\phi}}}{\tilde{\bar{\phi}}}-\frac{\mathbf{grad}\,\tilde{\bar{\phi}}^*}{\tilde{\bar{\phi}}^*}\right] +2|\tilde{\bar{\phi}}|^2\,\mathbf{grad}\,\beta\right\}. \quad (\mathrm{C}.5)$$

Furthermore, along the lines of reference [17], one can show that $\tilde{\bar{\phi}}$ has a negligible time evolution in the Thomas-Fermi limit; we can then replace $\tilde{\bar{\phi}}$ by its initial value $\bar{\phi}_0$ and we recover the first line of equation (47).

The second equation (C.4) is an acceptable approximation for the evolution of β if the neglected terms, all involving spatial derivatives of α, β or $\tilde{\bar{\phi}}$, are small as compared to the right hand side of equation (C.4), as they are expected to be in the Thomas-Fermi limit. Neglecting these terms, we recover the second line of equation (47).

In order to estimate the order of magnitude of the neglected terms in the time derivative of β, we calculate the exact derivative:

$$\begin{aligned}
\mathrm{i}\hbar\partial_t\beta = {} & \frac{1}{\lambda^3}N_\mathrm{a}(g_\mathrm{aa}-g_\mathrm{ab}) \\
& -\frac{\hbar^2}{2m\lambda^2}\,\mathbf{grad}\,\beta\cdot\left[\frac{\mathbf{grad}\tilde{\bar{\phi}}}{\tilde{\bar{\phi}}}-\frac{\mathbf{grad}\tilde{\bar{\phi}}^*}{\tilde{\bar{\phi}}^*}\right] \\
& -\frac{\hbar^2}{2m\lambda^2|\tilde{\bar{\phi}}|^2}\left\{\varDelta\alpha-\frac{1}{2}\,\mathbf{grad}\,\alpha\cdot\left[\frac{\mathbf{grad}\tilde{\bar{\phi}}}{\tilde{\bar{\phi}}}+\frac{\mathbf{grad}\tilde{\bar{\phi}}^*}{\tilde{\bar{\phi}}^*}\right]\right. \\
& \left.\qquad\qquad -\frac{1}{2}\alpha\left[\frac{\varDelta\tilde{\bar{\phi}}}{\tilde{\bar{\phi}}}+\frac{\varDelta\tilde{\bar{\phi}}^*}{\tilde{\bar{\phi}}^*}\right]\right\}\cdot
\end{aligned}\tag{C.6}$$

We replace $\tilde{\bar{\phi}}$ by $\overline{\phi_0}$. We consider an eigenmode with frequency $\varOmega_q$; from equation (52) we estimate $\varDelta\alpha/\alpha \sim q/R_0^2$. Assuming λ on the order of 1 we get the condition

$$\varOmega_q \gg q\frac{\hbar\omega^2}{\mu}\tag{C.7}$$

which we can rewrite as

$$1 \le \sqrt{q} \ll \frac{\varOmega_{q=1}}{\hbar\omega^2/\mu}\,\cdot\tag{C.8}$$

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19. Consider the case $\varOmega_q = \mathrm{i}\gamma$, where $\gamma > 0$; then A_q, B_q are real positive for all time and one has: $\frac{\mathrm{d}}{\mathrm{d}t}(A_qB_q) = \frac{\gamma}{\lambda^2}B_q^2 + \frac{\gamma}{\lambda^3}A_q^2 \ge \frac{\gamma}{\lambda_\mathrm{max}^3}(A_q^2+B_q^2) \ge \frac{2\gamma}{\lambda_\mathrm{max}^3}A_qB_b$ where $\lambda_\mathrm{max} = \mathrm{max}_t\lambda(t) \ge 1$.
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22. We do not get the linear divergence with time of the quadrupolar mode $q = 5$ predicted in [20] as the equations for $\delta\varphi_\mathrm{d}$ differ from the ones obtained for a single condensate, leading to a frequency of the breathing eigenmode $\varOmega_5$ very different from the one of a single condensate. The divergence of the quadrupolar mode occurs here for $\delta\varphi_\mathrm{s}$.
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24. More precisely we arrive at $\mathrm{div}[|\phi_0|^2\langle\delta\mathbf{v}_\varepsilon\rangle] = 0$. The velocity fields are of the form $f_\varepsilon(r)\mathbf{r}$ because the condensate wavefunctions are isotropic. Using Gauss formula we find that $f_\varepsilon \equiv 0$.
25. More precisely $\langle\delta\rho_\varepsilon\rangle$ and $\delta\rho_\varepsilon^\mathrm{st}$ differ by a constant; on the other hand this constant vanishes as both the density deviations have a vanishing integral over the volume of the condensate, so we conclude $\langle\delta\rho_\varepsilon\rangle = \delta\rho_\varepsilon^\mathrm{st}$.
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27. We have checked numerically this statement for the JILA case by looking at the time dependence of the relative distance between the two condensates; at long times this distance oscillates at a frequency close to an eigenfrequency of the steady state condensates.
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